MARIA PIRES DE CARVALHO

Chaotic Newton's Sequences

As a route to ever more exact knowledge, successive approximation has been a major theme in the development of science. Many algorithms to find approximations of roots of equations were devised. In all such reasonings we begin with an idea of where the root lies, albeit less than accurate, and we have a strategy to improve the estimates. To look up "whale" in a dictionary, the first step is to open the dictionary close to the end, because you have a rough idea where the word is; next, you turn the pages backward or forward till you find it, and this is the strategy to improve the first approximation. In the search for zeros of functions, you need to know that a zero exists and how the map behaves in the neighborhood of that zero.

Newton formulated a general and simple method to find approximations of zeros of functions. For a real (or complex) function f with a zero at ξ, and an initial choice x_0, Newton suggested the following recurrence formula to obtain better approximations of ξ:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

which is defined if the derivative of f vanishes at no x_n, and which, if convergent, will surely pick up a zero of f as its limit. Given x_0, the term x_1 is obtained by considering the tangent line at $(x_0, f(x_0))$ to the graph of f and intersecting it with the real axis; to get the whole sequence, just iterate this process. Sufficient conditions for the method to work are easy to state, but a major problem arises: the competition among the several zeros of the function. As a consequence, the basin of attraction of each zero (that is, the set of initial conditions x_0 such that the corresponding sequence $(x_n)_{n \in \mathbb{N}_0}$ converges to the specified zero) may have a very complicated boundary, and the dynamics associated to the sequences $(x_n)_{n \in \mathbb{N}_0}$ may be highly sensitive to perturbations on initial conditions. These boundaries have been a favorite showpiece in popularizing fractals (see for instance [DS]).

But here I will focus on another problem. What happens if a map $f: \mathbb{R} \to \mathbb{R}$ has no real zeros? Newton's sequences $(x_n)_{n \in \mathbb{N}_0}$ may be defined, although they will never converge. How do these sequences behave? I will examine here the particular case of the quadratic family $x \in \mathbb{R} \mapsto f_c(x) = x^2 + c$, where c is a real positive parameter. The natural extension to $\mathbb{C}$ of each map of the family has the real line $\mathbb{R} \times \{0\}$ as the boundary of the basins of attraction of its two (complex) roots, so its geometry is trivial. However, the sequences $(x_n)_{n \in \mathbb{N}_0}$ show irregular and unpredictable behavior, which nevertheless has an underlying order that I will describe.

After a clever change of variable, analysis of the sequences $(x_n)_{n \in \mathbb{N}_0}$ will be straightforward by appealing to some easy techniques and results from dynamical systems and elementary number theory. The main result is that rational initial conditions produce finite or infinite periodic sequences, whereas the irrational ones yield infinite but not periodic sequences. This recalls what happens with decimal or binary expansions (luckily, even the terminology is the same), and the sensitivity with respect to the initial choice x_0 is evinced at once. Moreover, the dynamics associated with these sequences is modeled by a left shift on the binary representation of x_0 in the new variable.

Let me start by taking a brief tour of discrete dynamical systems. Given a map $G: X \to X$, I may compose G with itself as many times as I please (the n-fold composition of G with itself is denoted by G^n). Therefore for each x in X the sequence $(G^n(x))_{n \in \mathbb{N}_0}$ is well defined; it is called *the or-*

bit of x by G. The set of all orbits is a *dynamical system*. Dynamical systems form a category in which an isomorphism between two dynamical systems $f: X \to X$ and $g: Y \to Y$ is given by a homeomorphism $h: X \to Y$ such that $g \circ h = h \circ f$; such an h is called a *conjugacy* between f and g. Essentially, the aim of the theory is to know, up to conjugacy, the asymptotic behaviors of each orbit and how they vary with x. The fixed points are the orbits easier to detect and the ones to look for first; more generally, an orbit is *periodic with period* $p \in \mathbb{N}$ if it is a fixed point of G^p; if nothing is said to the contrary, p is understood to be the smallest period. An orbit is *pre-periodic with pre-period* $n \in \mathbb{N}_0$ *and period* $p \in \mathbb{N}$ if $G^n(x)$ is a fixed point for G^p.

Rational numbers have finite or infinite periodic representations in any base.

For maps G defined on subsets of $\mathbb{R}$, the composition of G with itself may be pictured on the graph of G, and this is a good way of guessing how the orbits behave. For instance, consider $G: [0, 1] \to [0, 1]$ given by $G(x) = 1 - x$. Then $G(x) = x$ if and only if $x = \frac{1}{2}$, for this is the only intersection of the graphs of G and the identity map. If $x \neq \frac{1}{2}$, then $G^2(x) = G(1 - x) = x$, so the orbit of x is periodic with period 2: I suggest you check this on the graph of G.

The orbits may present many differences with respect to their topological properties, asymptotic behavior, or cardinality of their range of values. There are dynamical systems that contain essentially all the kinds of orbits that non-injective maps may be expected to have. One such system is based on the space of sequences constructed with the digits 0 and 1, say $\Sigma = \{0, 1\}^{\mathbb{N}} = \{(a_1, a_2, \cdots, a_n, \cdots) : a_j \in \{0, 1\}\}$, with the metric

$$D(z, w) = \sum_{j=1}^{\infty} \frac{|a_j - b_j|}{2^j},$$

for $z = (a_1, a_2, \cdots, a_n, \cdots)$ and $w = (b_1, b_2, \cdots, b_n, \cdots)$. Acting on Σ, the one-sided full shift map σ takes each sequence $(a_1, a_2, \cdots, a_n, \cdots)$ to $(a_2, \cdots, a_n, \cdots)$. This map is continuous with respect to the above metric; it has periodic points of all periods, because, for each $p \in \mathbb{N}$,

$$\sigma^p(a_1, a_2, \cdots, a_p, a_1, a_2, \cdots, a_p, \cdots\cdots) = (a_1, a_2, \cdots, a_p, a_1, a_2, \cdots, a_p, \cdots\cdots);$$

and it has dense orbits (e.g., that of the element of Σ that is obtained by writing down consecutively all possible finite blocks of digits 0 or 1 ordered by their length—see [D] for more details). I will consider each element of Σ as a binary expansion of a number in [0, 1]; in this process, the finite binary representation (of each dyadic rational) is thought of as having an infinite tail of zeros: thus, $0.01_{(2)}$ is the element of Σ given by $01000000\cdots$, and is distinct, in Σ, from $00111111\cdots$, although they are expansions of the same number.

In expansion of the real numbers in a given base b, each number is replaced by a sequence $a_N \cdots a_0 \cdot c_1c_2c_3 \cdots c_k \cdots\cdots_{(b)}$ with a_j, c_k in $\{0, 1, \cdots, b-1\}$, meaning that the number is given by the sum

$$a_N(b)^{N-1} + \cdots + a_1b + a_0 + c\left(\frac{1}{b}\right) + \cdots + c_k\left(\frac{1}{b}\right)^k + \cdots\cdots$$

It will be found useful to discard the integer part and keep information only about the digits c_k. Rational numbers have finite or infinite periodic representations in any base, in general not unique; irrationals appear as unique non-periodic infinite representations. To simplify the notation, a periodic sequence of Σ, say $(a_1, a_2, \cdots, a_p, a_1, a_2, \cdots, a_p, \cdots\cdots)$, will be denoted by $\overline{a_1a_2\cdots a_p}$, and similarly a pre-periodic binary representation $0.a_1a_2 \cdots a_na_{n+1}a_{n+2} \cdots a_{n+p}a_{n+1}a_{n+2} \cdots a_{n+p} \cdots\cdots$ will be abbreviated to $0.a_1a_2\cdots a_n\,\overline{a_{n+1}\,a_{n+2}\cdots a_{n+p}}$.

When a rational number is written in irreducible form, information on its expansion in a given base can be read from the denominator only. In the case $b = 2$ it is known that (see [RT]):

(I) *A rational $r_0 \in]0, 1[$ has finite binary representation if and only if it is dyadic; that is, it may be written as $r_0 = k/2^n$ where k, $n \in \mathbb{N}$ and k is odd.*

In this case the (finite) representation of r_0 has precisely n digits.

(II) *A rational $r_0 \in]0, 1[$ has infinite binary representation with a period that starts just after the decimal point if and only if it is an irreducible fraction t/q where q is odd.*

Furthermore, the length of the period does not exceed $\phi(q)$, where ϕ is the Euler totient function (for each $q \in \mathbb{N}$, $\phi(q)$ is the number of positive integers less than q and co-prime to q); in fact, it divides ϕ(denominator) and is independent of the numerator. (For instance, $1/5 = 0.\overline{0011}_{(2)}$ has period $4 = \phi(5)$ and $1/13 = 0.\overline{000100111011}_{(2)}$ has period $12 = \phi(13)$.)

(III) *The denominator is even but not a power of 2—that is, $r_0 = t/2^nQ$, an irreducible fraction where Q is odd and n is a positive integer—if and only if the binary representation is infinite pre-periodic with a pre-period n.*

For example, $1/(2 \cdot 5) = 0.0\overline{0011}_{(2)}$ has period 4 as 1/5 and pre-period 1.

Cases (II) and (III) merit closer inspection:

(IV) *If an irreducible fraction of positive integers $t/q \in]0, 1[$ has an odd denominator, it may be expressed in the form $s/(2^p - 1)$ where s and p are positive integers and are minimal. Once this is achieved, p gives the length of the period of its binary representation.*

For example,

$$\frac{1}{5} = \frac{3}{2^4 - 1} = 0.\overline{0011}_{(2)};\ \frac{1}{13} = \frac{5 \times 63}{2^{12} - 1} = 0.\overline{000100111011}_{(2)}.$$

(V) *If the fraction t/q has an even denominator which is not a power of 2—that is, $t/q = t/2^nQ$ with $n \in \mathbb{N}$ and Q odd—it may be expressed in the form $s/2^n(2^p - 1)$ where n, s, and p are positive integers, minimal, and p is greater than 1. The integer p is the length of the period of the binary representation of t/q, and n is the pre-period.*

For example $1/12 = 1/(2^2\,(2^2 - 1)) = 0.00\overline{01}_{(2)}$.

Let me sketch a proof of these two properties. (V) implies (IV) if n is also allowed to be zero; to prove (V), consider the fraction $1/Q$ and the equations that produce its binary expansion:

$$\begin{array}{lll} 1 & = Q \times 0 + 1 & \\ 2 \times 1 & = Q \times d_1 + r_1 & 0 < r_1 < Q \\ 2 \times r_1 & = Q \times d_2 + r_2 & 0 < r_2 < Q \\ \vdots & & \end{array}$$

As the remainders r_j are positive integers less than and coprime to Q, they repeat themselves after $\phi(Q)$ steps, at the most. The first remainder to reappear is precisely 1 because, by (II), the binary representation of $1/Q$ has a period that starts just after the decimal point. Therefore there exists a positive index p such that $r_p = 1$, and so the last of the above equations, before they start repeating, is $2 \times r_{p-1} = Q \times d_p + r_p = Q \times d_p + 1$. Multiply the second equation by 2^{p-1}, the third one by 2^{p-2} and so on, and add them all to get

$$2^p = Q\,[2^{p-1}\,d_1 + 2^{p-2}\,d_2 + \cdots + 2d_{p-1} + d_p] + 1.$$

Therefore

$$\frac{1}{Q} = \frac{[2^{p-1}\,d_1 + 2^{p-2}\,d_2 + \cdots + 2d_{p-1} + d_p]}{2^p - 1} = \frac{A}{2^p - 1},$$

so

$$\frac{t}{Q} = \frac{At}{2^p - 1},$$

$$\frac{t}{2^nQ} = \frac{At}{2^n\,(2^p - 1)} = \frac{s}{2^n\,(2^p - 1)}.$$

Further, the type of the binary representation of $s/(2^n\,(2^p - 1))$ is the same as that of $1/(2^n\,(2^p - 1))$, and the latter may be obtained from the following calculation:

$$\frac{1}{2^n\,(2^p - 1)} = \frac{1}{2^n}\,\frac{1/2^p}{1 - 1/2^p}$$
$$= \frac{1}{2^n}\sum_{j=1}^{\infty}\left(\frac{1}{2^p}\right)^j = 0.0\cdots0\overline{00\cdots01}_{(2)},$$

where the first block of zeros has size n and the repeating block has $p - 1$ zeros followed by a single 1. The integer s may change the digits but not the meaning of n and p. Notice that if the denominator is even but not a power of 2, then p must be bigger than or equal to 2. The effect of the power 2^n in the denominator is to push the period to the right, creating a pre-period of length n. I suggest you check this on some examples, such as

$$\frac{1}{14} = \frac{1}{2(2^3 - 1)} = 0.0\overline{001}_{(2)};\ \frac{9}{14} = \frac{9}{2(2^3 - 1)} = 0.1\overline{010}_{(2)};$$

$$\frac{1}{28} = \frac{1}{2^2\,(2^3 - 1)} = 0.00\overline{001}_{(2)}.$$

Let me summarize for later use:

$$\begin{cases} r_0 \notin \mathbb{Q} \Rightarrow r_0 \text{ has a unique representation, infinite, non-periodic} \\ r_0 \in \mathbb{Q} \Rightarrow \begin{cases} \exists k, n \in \mathbb{N} : r_0 = k/2^n \Rightarrow \text{has a finite binary representation that terminates at 0 after } n \text{ digits} \\ \exists k, n \in \mathbb{N}\ \exists p \in \mathbb{N}_0 : r_0 = k/(2^p(2^n - 1)) \Rightarrow \text{unique binary representation with pre-period } p \text{ and period } n \end{cases} \end{cases}$$

It is time to go back to Newton's method and the map f_1. If I start with an initial condition $x_0 \in \mathbb{R}$, then the corresponding Newton's sequence $(x_n)_{n\in\mathbb{N}_0}$, if well defined, is real and thus cannot converge: if it did, the recurrence formula $x_{n+1} = (x_n^2 - 1)/2x_n$ would imply that the limit $L \in \mathbb{R}$ verifies the impossible equation $2L^2 = L^2 - 1$. The dynamical system associated with this recurrence formula may be described by the iterates of the map $\mathscr{G} : \mathbb{R} \to \mathbb{R}$, $\mathscr{G}(t \neq 0) = (t^2 - 1)/2t$, $\mathscr{G}(0) = 0$. If well defined, the sequence $(x_n)_{n\,\in\,\mathbb{N}_0}$ is the orbit by $\mathscr{G}$ of x_0; however, once an orbit of $\mathscr{G}$ lands on the fixed point 0, it stops being a Newton's sequence. The map $\mathscr{G}$ is an odd function, increasing in $]-\infty, 0[$ and in $]0, +\infty[$, and is asymptotic to the line $y = x/2$. It is easy to identify some orbits by observing the graph of $\mathscr{G}$:

(1) Consider $x_0 = 1$; then $\mathscr{G}(x_0) = 0$, so $\mathscr{G}^n(x_0) = 0$ for $n \geq 1$; x_n is not defined for $n \geq 2$. I describe this by saying that the orbit of 1 is *finite and terminates at* 0 *after one iterate.*

(2) If $x_0 = 1 + \sqrt{2}$, then $\mathscr{G}(x_0) = 1$ and $\mathscr{G}^2(x_0) = 0$, so $\mathscr{G}^n(x_0) = 0$ for $n \geq 2$ although x_n is not defined for $n \geq 3$. This orbit is also finite and terminates at 0 after two iterates.

(3) Take now $x_0 = 1/\sqrt{3}$; then $\mathscr{G}(x_0) = -1/\sqrt{3}$ and $\mathscr{G}^2(x_0) = 1/\sqrt{3}$. This is a periodic orbit of period two. The equality $\mathscr{G}^2(x) = x$ leads to a polynomial equation of degree 4 with only even exponents; it has no solutions other than $1/\sqrt{3}$ and $-1/\sqrt{3}$.

(4) If $x_0 = \sqrt{3}$, then $\mathscr{G}(x_0) = 1/\sqrt{3}$ and $\mathscr{G}^2(\mathscr{G}(x_0)) = \mathscr{G}(x_0)$. So x_0 is a pre-periodic orbit of period two and pre-period one.

More sophisticated tools are needed to detect other kinds of orbit. The recurrence formula $x_{n+1} = ((x_n)^2 - 1)/(2x_n)$ is similar to the trigonometric formula $\text{cotan}(2\theta) = (\text{cotan}^2(\theta) - 1)/(2\,\text{cotan}(\theta))$ for $\theta \in\,]0, \pi[\,/\,\{\pi/2\}$. Let $x_0 = \text{cotan}(\pi r_0)$ for $r_0 \in\,]0, 1[$: this is permissible since cotan: $]0, \pi[\, \to \mathbb{R}$ is a homeomorphism, and so the topological properties of the orbits of $\mathscr{G}$ are preserved under this change of variable. Moreover, in this notation, we have $\mathscr{G}^n\,(x_0) = \text{cotan}(\pi 2^n r_0)$ for each n, provided that $2^n\pi r_0$ is not an integer multiple of π. The numbers in $]0, 1[$ that fail to satisfy this requirement for some integer n are just the dyadic rationals; more precisely:

1st Conclusion: $r_0 = k/2^n$, *with* $k, n \in \mathbb{N}$ *and* k *an odd integer, if and only if the orbit by* $\mathscr{G}$ *terminates at* 0 *after* n *iterates.*

Because k is odd, we have $x_{n-1} = \text{cotan}(\pi 2^{n-1} r_0) = \text{cotan}(\pi k/2) = 0$ and therefore x_m is not defined for $m \geq n$; so the orbit of $x_0 = \text{cotan}(\pi r_0)$ by $\mathscr{G}$ terminates at the fixed point 0 after n iterates. This is the case of $r_0 = 1/4 = 0.01_{(2)}$, $x_0 = \text{cotan}(\pi r_0) = \text{cotan}(\pi/4) = 1$ and $x_1 = 0$. Conversely, if an orbit of $\mathscr{G}$ terminates at 0, say $\mathscr{G}^n(x_0) = 0$, then $\text{cotan}(\pi 2^n r_0) = 0$ and therefore there exists $m \in \mathbb{Z}$ such that $2^n \pi r_0 = m\pi + \pi/2$. So $2^n r_0 = m + 1/2$, that is, $r_0 = (2m+1)/2^{n+1}$.

What real numbers r_0 produce periodic or pre-periodic orbits by $\mathscr{G}$? r_0 cannot be dyadic, and there must be N and P such that $\mathscr{G}^{N+P}(x_0) = \mathscr{G}^N(x_0)$; this implies that $r_0 \neq k/2^n$ for all integers k and n and $\text{cotan}(\pi 2^{N+P} r_0) = \text{cotan}(\pi 2^N r_0)$. Solving this equation, it is found that $r_0 = k/2^N(2^P - 1)$ with $k \in \mathbb{N}, N \in \mathbb{N}_0, P \in \mathbb{N}$ and $P \geq 2$. These are the remaining rationals of $]0, 1[$ (see (IV) and (V) above): they have infinite periodic or pre-periodic binary expansions with period P.

2nd Conclusion: *The orbit of* x_0 *by* $\mathscr{G}$ *is finite or infinite periodic/pre-periodic if and only if* r_0 *is rational; if such is the case, then the orbit type of* x_0 *is completely determined by the denominator of* r_0. *In particular, if* r_0 *is irrational, then* x_n *is defined for all* $n \in \mathbb{N}$.

Let me review in this new setting some of the above examples.

(a) $r_0 = 1/3 = 1/(2^{-2} - 1) = 0.\overline{01}_{(2)}$: then $N = 0, P = 2, x_0 = \text{cotan}(\pi/3) = 1/\sqrt{3}$, and $x_1 = \text{cotan}(2\pi/3) = -1/\sqrt{3}$. The orbit by $\mathscr{G}$ of x_0 is periodic with period P.

(b) $r_0 = 1/6 = 1/2(2^2 - 1) = 0.0\overline{01}_{(2)}$: $N = 1$, $P = 2$, and $x_0 = \text{cotan}(\pi/6) = \sqrt{3}, x_1 = \text{cotan}(2\pi/6) = \text{cotan}(\pi/3) = 1/\sqrt{3}$. The orbit of x_0 is pre-periodic with pre-period $N = 1$ and period $P = 2$.

(c) $r_0 = 1/5 = 3/(2^{-4} - 1) = 0.\overline{0011}_{(2)}$: $N = 0, P = 4$, and $x_0 = \text{cotan}(\pi/5)$, $x_1 = \text{cotan}(2\pi/5)$, $x_2 = \text{cotan}(4\pi/5)$, $x_3 = \text{cotan}(8\pi/5)$, $x_4 = \text{cotan}(16\pi/5) = x_0$. The orbit of x_0 is periodic with period $P = 4$.

I suggest you now compare the following diagram with the similar one above.

$$\begin{cases} r_0 \notin \mathbb{Q} \Rightarrow \text{its orbit by } \mathscr{G} \text{ is infinite non-periodic} \\ r_0 \in \mathbb{Q} \Rightarrow \begin{cases} \exists k, n \in \mathbb{N} : r_0 = k/2^n \Rightarrow \text{its orbit by } \mathscr{G} \\ \quad \text{terminates at 0 after } n \text{ iterations} \\ \exists k, n \in \mathbb{N} \ \exists p \in \mathbb{N}_0 : r_0 = k/2^p(2^n - 1) \Rightarrow \text{its} \\ \quad \text{orbit by } \mathscr{G} \text{ has pre-period } p \text{ and period } n \end{cases} \end{cases}$$

Thus the orbit of x_0 by $\mathscr{G}$ is completely determined by the binary representation of r_0. This also shows that the discrete dynamical system generated by $\mathscr{G}$ is highly sensitive to initial conditions: the distinction between rational and irrational r_0 is enough to produce wide disparities between orbits.

Other more particular traits of the orbits for irrational values of r_0 can be studied by picking up two clues I left behind:

(1) the function $z \mapsto \text{cotan}(\pi z)$ is periodic of period 1;

(2) iterating x_0 by $\mathscr{G}$ corresponds, in the new variable, to simply doubling the argument of the cotan function.

The first one implies that, when you compute the successive values of $\text{cotan}(\pi 2^n r_0)$, what matters is the fractional part of $2^n r_0$ (denoted by $\{2^n r_0\}$). If the irrational r_0 is written in base 2 as $r_0 = 0\,.\,a_1 a_2 a_3 \cdots a_k \cdots_{(2)}$, this representation is unique, and $2r_0 = a_1\,.\,a_2 a_3 \cdots a_k \cdots_{(2)}$. Dismissing the integer part, we are left with $\{2r_0\} = 0.a_2 a_3 \cdots a_k \cdots_{(2)}$ and, by (2),

$$\begin{aligned}(\text{cotan}(\pi 2^n r_0))_{n \in \mathbb{N}_0} &= (\text{cotan}(\{\pi 2^n r_0\}))_{n \in \mathbb{N}_0} \\ &= (\text{cotan}(\pi \cdot 0\,.\,a_{n+1} a_{n+2} \cdots_{(2)}))_{n \in \mathbb{N}_0},\end{aligned}$$

which corresponds, up to the action of $\text{cotan} \circ (\pi \times \cdot\,)$, exactly to the iteration σ^n of the shift on the sequence $a_1 a_2 a_3 \cdots a_k \cdots$. More precisely, the map

$$\begin{aligned}]0, 1[\, / \{\text{dyadic numbers}\} &\xrightarrow{\mathscr{T}} \,]0, 1[\, / \{\text{dyadic numbers}\} \\ 0\,.\,a_1 a_2 \cdots a_k \cdots_{(2)} &\mapsto 0\,.\,a_2 a_3\,.\,a_k \cdots\cdots_{(2)} \end{aligned}$$

(that is, $\mathscr{T}(t) = 2t$ if $0 \leq t < \frac{1}{2}$, $\mathscr{T}(t) = 2t - 1$ if $\frac{1}{2} \leq t < 1$) is conjugated by $z \mapsto \text{cotan}(\pi z)$ to the action of $\mathscr{G}$ on the set of x_0 whose orbits by $\mathscr{G}$ do not terminate at the fixed point 0 after a finite number of iterates; and $\mathscr{T}$ is the same as the shift map σ restricted to the sequences of zeros or ones that are not eventually constant, for the map

$$h(0\,.\,a_1 a_2 \cdots a_k \cdots_{(2)}) = a_1 a_2 a_3 \cdots a_k \cdots\cdots$$

is a conjugacy between the chosen restrictions of $\mathscr{T}$ and σ.

Let me illustrate the use of these observations in two examples:

(i) If $r_0 = 0.10100100010000 \cdots\cdots_{(2)}$, where each digit 1 is followed by a block of zeros of increasing length, then r_0 is irrational and the sequence $(x_n)_{n \in \mathbb{N}} = (\text{cotan}(\pi\, 2^n r_0))_{n \in \mathbb{N}} = (\text{cotan}(\{\pi 2^n r_0\}))_{n \in \mathbb{N}}$ is bounded away from zero, because $\{2^n r_0\} < 0.1010010010 \cdots_{(2)} = \frac{9}{14}$ for all n. But, since $\{2^n r_0\}$ gets arbitrarily close to 0, this orbit is not bounded from above.

(ii) If r_0 is an irrational number whose binary representation is given by a sequence in Σ with dense σ-orbit, then the corresponding sequence $(x_n)_{n \in \mathbb{N}_0}$ is dense in $\mathbb{R}$.

If for each dyadic number of $]0, 1[$ I select the binary representation with ending zeros (e.g., writing $1/2 = 0.10000 \cdots_{(2)}$ instead of $0.0111 \cdots_{(2)}$), then the corresponding extension of h is not continuous. However, if I let $\mathscr{H}(x) = h((1/\pi)\text{cotan}^{-1} x)$, then the equation $\sigma \circ \mathscr{H}(x) = \mathscr{H} \circ \mathscr{G}(x)$ is still valid for all $x \neq 0$. This yields the following:

3rd Conclusion: *The dynamics of the Newton's sequences* $(x_n)_{n \in \mathbb{N}_0}$, *for allowed real initial conditions* x_0, *is determined by the binary representations of the initial conditions in the new variable* r_0.

I now proceed to check how the parameter c affects the previous calculations. I will show that the dynamics of the corresponding Newton's sequences for parameter c is the same as for $c = 1$ when $c > 0$, and changes drastically at $c = 0$.

Let me rewrite c as $\pm a^2$, with $a \in [0, +\infty[$. Denote by $\mathcal{G}_a^{\pm}$ the map associated to Newton's method applied to f_c, where $\pm = \text{sign}(c)$: thus $\mathcal{G}_a^{\pm}(0) = 0$, $\mathcal{G}_a^+(x) = (x^2 - a^2)/2x$, $\mathcal{G}_a^-(x) = (x^2 + a^2)/2x$. For a fixed sign $\pm$, the family of maps $(\mathcal{G}_a^{\pm})_{a \in]0, +\infty[}$ converges pointwise, but not uniformly, to $\mathcal{G}_0(x) = x/2$ as $a \to 0$. The limiting dynamics is uninteresting: for all $x_0 \in \mathbb{R}$, the sequence $((\mathcal{G}_0)^n(x_0))_{n \in \mathbb{N}}$ has limit 0, the unique fixed-point of $\mathcal{G}_0$.

If $a > 0$, then for $x \neq 0$ we have

$$\mathcal{G}_a^+(x) = \frac{x^2 - a^2}{2x} = a\frac{\left(\frac{x}{a}\right)^2 - 1}{2\left(\frac{x}{a}\right)},$$

that is,

$$\frac{\mathcal{G}_a^+(x)}{a} = \frac{\left(\frac{x}{a}\right)^2 - 1}{2\left(\frac{x}{a}\right)}.$$

This suggests the change of variable

$$t_0 = \frac{x_0}{a},$$

which leads to

$$t_1 = \frac{x_1}{a} = \frac{(t_0)^2 - 1}{2t_0}$$

and, in general, to

$$t_{n+1} = \frac{(t_n)^2 - 1}{2\, t_n}.$$

This means that, up to a change of variable, the map $\mathcal{G}_a^+$ acts as $\mathcal{G} = \mathcal{G}_1^+$, and no further work is needed in this case.

If $a > 0$ and $c = -a^2$, then f_c has two real zeros, a and $-a$, with basins of attraction given by $]0, +\infty[$ and $]-\infty, 0[$, respectively. In fact, the minimum value of $\mathcal{G}_a^-(x) = x^2 + a^2/2x$ for $x > 0$ is a, which is also the unique fixed point of $\mathcal{G}_a^-$ in $]0, +\infty[$; and, since $\mathcal{G}_a^-|_{[a, +\infty[}$ is a contraction, it follows that, for all initial choices $x_0 > 0$, the sequence $(x_n)_n$ converges to a. Similar reasoning shows that $(x_n)_n$ converges to $-a$ for all choices $x_0 < 0$. It is along the imaginary axis that the dynamics of $\mathcal{G}_a^-$ is chaotic: for, if $x_0 = i\rho_0$ for some $\rho_0 \in \mathbb{R} / \{0\}$, then Newton's recurrence formula $x_{n+1} = (x_n{}^2 + a^2)/2\, x_n$ becomes

$$i\rho_{n+1} = \frac{(i\, \rho_n)^2 + a^2}{2\, i\rho_n} = i\frac{(\rho_n)^2 - a^2}{2\rho_n}.$$

This means that, in the real variable ρ, the dynamics is given by $\rho_{n+1} = \mathcal{G}_a^+(\rho_n)$, which has already been analyzed.

It is worth remarking that the conclusions obtained for the quadratic family $(f_c)_c$ extend easily to all quadratic polynomials. Given a polynomial $p(x) = d_2x^2 + d_1x + d_0$, with $d_j \in \mathbb{R}$ and $d_2 \neq 0$, the equation $p(x) = 0$ is equivalent to $p(x)/d_2 = 0$, and so I may assume that $d_2 = 1$. By a simple translation in the variable x, given by $x = t + d_1/2$, p becomes

$$p(t) = t^2 + [d_0 - d_1^2/4],$$

which belongs to the family $(f_c)_c$. Hence all the previous results hold for this larger family.

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AUTHOR

MARIA PIRES DE CARVALHO

Centro de Matemática do Porto

Praça Gomes Teixeira

4099-002 Porto

Portugal

e-mail: mpcarval@fc.up.pt

Maria Carvalho and her twin sister were born in Africa. She completed her first degree in mathematics at the University of Porto, where she is now an associate professor. Her postgraduate studies were completed at Instituto de Matemática Pura e Aplicada, in Rio de Janeiro, where she specialized in Ergodic Theory and completed her Ph.D. under the guidance of Ricardo Mañé. Maria shares a cat with her husband and is enthusiastic about literature and jazz music.